K-Color Ramsey Theorem Tighter Bound

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1 Background

Ramsey's Theorem deals with the minimum value n such that any K-coloring of the edges of complete graph K_n contains a clique of size t where all edges are the same color. A generalization of this min value $R_k(t)$ is $R_k(v)$ where v is the k-dimensional vector $(v_1, v_2, ..., v_k), v_i \geq 1$ defined by

$$R_k(v) = min(n) \ st \ \forall \ k\text{-}colorings \ of \ K_n \ \exists \ clique \ of \ color \ i \ of \ size \ v_i$$

There have been several very loose bounds for the general value of a k-coloring: the tightest in literature is found in a 1955 paper by Greenwood and Gleeson that determined that

$$R_k(v_1, v_2, ..., v_k) \leq R_k(v_1-1, v_2, ..., v_k)+R_k(v_1, v_2-1, ...v_k)+...R_k(v_1, v_2, ...v_k-1)$$

which provided the bound of a multinomial coefficient bound of

$$R_k(v) \leq \frac{(v_1 + v_2 ... + v_k)!}{v_1!v_2!...v_k!}$$

and for the specific case of $v = (t, t, t...)$ this reduces to

$$R_k(v) \leq \frac{(kt)!}{t!^k} \tag{1}$$

However I now present a proof for a stricter bound of

$$R_k(v) \leq k^{\sum_{i=1}^{k}(v_i-1)} \tag{2}$$

which reduces in the specific case of $v = (t, t, t...)$ to

$$R_k(v) \leq k^{k(t-1)}$$

which is stronger than known bounds where $k > e * t$

2 Proof

The proof of one proceeds in a similar vein to the original, based on induction on the sum of the elements of v, $\sigma(v) = \sum_{i=1}^{k} v_i$.
Base case: For given v $\exists i \mid v_i = 1$. Then a graph of one node would suffice since any coloring would have a clique of size one where all edges (empty set) are colored v_i. This trivially fits the inequality since $\sigma(v) \geq k$.

$$k^{(\sigma(v)-k)} \geq k^{(k-k)} = 1 = R_k(v)$$

Inductive Step: Assume this holds true *crall* v where $\sigma(v) \leq n$. Now to prove for the case where $\sigma(v) = n+1$.
Let D be a k-coloring of K_n where $n = k^{\sigma(v)-k}$, $\sigma(v) = n+1$. Pick an arbitrary node a. Using the extended pigeon-hole principle, which states that if one has x items to put into y boxes then $\exists$ a box with $\lceil \frac{x}{y} \rceil$ to this problem, we get that at least $\lceil \frac{k^{\sigma(v)-k}-1}{k} \rceil = k^{\sigma(v)-k-1}$ of the edges to nodes coming from a are colored with color c $\in$ [n]. Now consider the subgraph S generated by those nodes. Define v' as follows:

$$v'_i = \begin{cases} v_i - 1, & \text{if } i = c \\ v_i, & \text{otherwise} \end{cases}$$

Thus $\sigma(v') = \sigma(v) - 1$, and by the inductive hypothesis we have a subgraph S of $k^{\sigma(v')-k}$ nodes and therefore it must contain a clique at least of size v'_i where all the edges are colored using color i.
There are 2 cases:

Case 1 - S contains a clique C of size $v_i, i \neq c$ with edges colored i. Then we are done since K_n would contain a clique of size v_i of color i.
Case 2 - S contains a clique C of size $v_i - 1, i = c$ with edges colored c. Then the Graph $a \cup C$ would be a clique of size v_i since every $\exists$ edge from a of color c to every node in the clique C. This means K_n would contain a clique of size v_i of color i.
In either case, K_n contains a clique of size v_i with edges colored i, completing the proof.

3 Comparative Analysis

The advantage of this style of proof over the original one in 1955 is that induction on the sum leads to more powerful results than an induction over all distinct tuples of the elements. Specifically, in the case where $v = (t, t, t, ...)$ the ratio of the two different bounds ends up being

$$R = \frac{t!^k * k^{k(t-1)}}{(kt)!} \tag{3}$$

Now we show that $R < 1$ holds true for $k > e * t$

Analysis relies heavily on use of factorial inequality

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}} \tag{4}$$

so

$$R \leq e^{k-1} * \frac{t^k}{k^k} < (\frac{et}{k})^k$$

which means $R < 1$ when $k > e * t = O(t)$

Interestingly, this doesn't seem to be the strongest bound. Analysis shows that $R < 1$ when $k = t$ as well.